

Pearson Edexcel Level 3

GCE Mathematics

Advanced

Paper 2: Pure Mathematics

PMT Mock 2

Time: 2 hours

Paper Reference(s)

9MA0/02

You must have:

Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 14 questions in this paper. The total mark is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.



Answer ALL questions.

1. The figure 1 shows part of the graph of $y = f(x)$, where $f(x) = \frac{ax+4}{x-b}$, $x > 2$

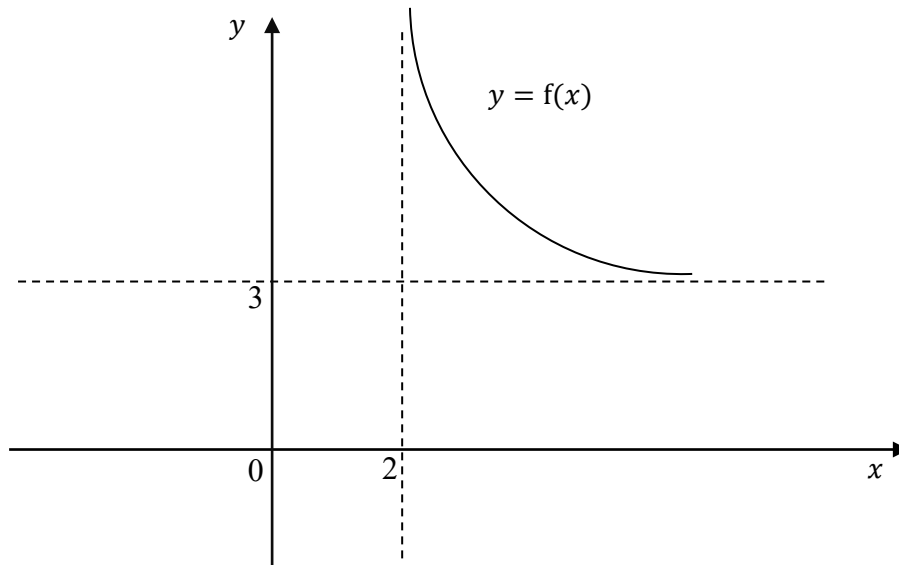


Figure 1

a. State the values of a and b .

B1 $b = 2$

B1 $a = 3$

(2)

b. State the range of f .

B1 $f(x) > 3$ or $y > 3$

(1)



c. Find $f^{-1}(x)$, stating its domain.

$$y = \frac{2x+4}{x-2}$$

$$x = \frac{2y+4}{y-2}$$

$$x(y-2) = 3y-4$$

$$xy - 2x = 3y - 4$$

$$xy - 3y = 2x - 4$$

$$y(x-3) = 2x-4$$

$$y = \frac{2x-4}{x-3}, x > 3$$

M1 Attempts at the method for finding the inverse

Look for a minimum of cross multiplying by $(x-2)$ and proceeding to form a

A1 Correct inverse $f^{-1}(x) = \frac{2x+4}{x-3}$, or $y = \frac{2x+4}{x-3}$

A1 Correct domain $x > 3$

(3)

(Total for Question 1 is 6 marks)



2. Relative to a fixed origin O ,

the point A has position vector $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

the point B has position vector $(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

and the point C has position vector $(-\mathbf{i} + \mathbf{j} + a\mathbf{k})$, where a is a constant and $a > 0$.

Given that $|\overrightarrow{BC}| = \sqrt{41}$

a. show that $a = 2$.

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} \quad \Rightarrow \quad \overrightarrow{BC} = (-\mathbf{i} + \mathbf{j} + a\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$

$$\overrightarrow{BC} = -2\mathbf{i} - \mathbf{j} + (a + 4)\mathbf{k}$$

$$(-2)^2 + (-1)^2 + (a + 4)^2 = 41$$

$$(a + 4)^2 = 36$$

$$a = 2 \quad \text{or} \quad a = -10$$

From the question, $a > 0$, so we reject the solution $a = -10$ and have $a = 2$

M1 Finds the difference between \overrightarrow{OC} and \overrightarrow{OB} then squares and adds each of the 3 components

M1 Complete method of correctly applying Pythagoras Theorem on $|\overrightarrow{BC}| = \sqrt{41}$ and using a correct method of solving their resulting quadratic equation to find at least one of $a = \dots$

A1 Obtains only one value, $a = 2$, $a > 0$

(3)



D is the point such that $ABCD$ forms a parallelogram.

b. Find the position vector of D .

$$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$$

$$\overrightarrow{BC} = \overrightarrow{AD} = -2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$-2\mathbf{i} - \mathbf{j} + 6\mathbf{k} = \overrightarrow{OD} - (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$\overrightarrow{OD} = \mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$$

M1 Complete applied strategy to find a vector expression for \overrightarrow{OD}

A1 Correct position vector for $\overrightarrow{OD} = \mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$

(2)

(Total for Question 2 is 5 marks)



3. a. “If p and q are irrational numbers, where $p \neq q, q \neq 0$, then $\frac{p}{q}$ is also irrational.”

Disprove this statement by means of a counter example.

M1 States or uses any pair of different numbers that will disprove the statement.

e.g. $\sqrt{2}, \sqrt{8}$; $\sqrt{3}, \sqrt{12}$; $\pi, 2\pi$; $3e, 7e$

e.g. $p = \sqrt{2}, q = \sqrt{8} \quad \frac{p}{q} = \frac{\sqrt{2}}{\sqrt{8}} \Rightarrow \frac{\sqrt{2}}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = \frac{1}{2}$

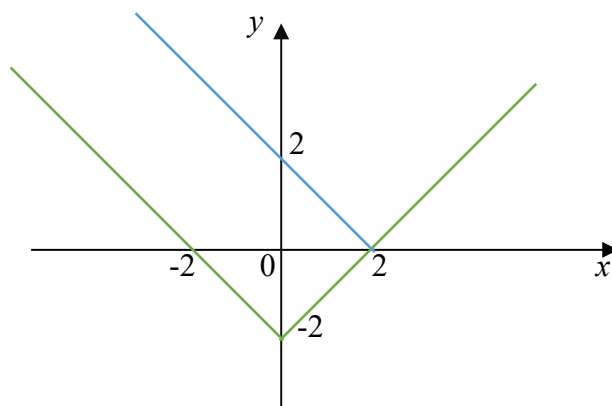
\Rightarrow statement untrue or $\frac{1}{2}$ is not irrational or $\frac{1}{2}$ is rational

A1 Uses correct reasoning to disprove the given statement, with a correct conclusion

(2)

b. (i) Sketch the graph of $y = |x| - 2$.

(ii) Explain why $|x - 2| \geq |x| - 2$ for all real values of x .



B1 V shaped graph symmetrical about the y-axis, with intercepts (2,0), (0,-2) and (2,0) (see green graph)

M1 draws the graph of $y = |x - 2|$ on top of the graph of $y = |x| - 2$ (see blue graph)

A1 the graph of $y = |x - 2|$ is either the same or above the graph of $y = |x| - 2$

or when $x \geq 0$, the graph of $y = |x - 2|$ is above the graph of $y = |x| - 2$

(3)

(Total for Question 3 is 5 marks)



4. (a) Show that $\sum_{r=1}^{20} (2^{r-1} - 3 - 4r) = 1047675$

$$\begin{aligned} \sum_{r=1}^{20} (2^{r-1} - 3 - 4r) &= \sum_{r=1}^{20} 2^{r-1} - \sum_{r=1}^{20} (3 + 4r) \\ &= 2^0 + 2^1 + 2^2 + \dots + 2^{19} - (7 + 10 + 13 + \dots + 83) \end{aligned}$$

The sequence $2^0 + 2^1 + 2^2 + \dots + 2^{19}$ is a geometric progression with $a = 1, r = 2$. The sum of the sequence is given by:

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{20} &= \frac{1(2^{20} - 1)}{2 - 1} = 1048575 \end{aligned}$$

The sequence $7 + 10 + 13 + \dots + 83$ is an arithmetic progression with $a = 7, d = 3$. The sum of the sequence is given by:

$$\begin{aligned} S_n &= \frac{1}{2}n(a + l) \\ S_{20} &= 10(7 + 83) = 900 \end{aligned}$$

Therefore:

$$\sum_{r=1}^{20} (2^{r-1} - 3 - 4r) = 1048575 - 900 = 1047675$$

M1 uses a correct method to find the given sum

M1 Correct method for finding the sum of a geometric progression

M1 Correct method for finding the sum of an arithmetic progression

A1 Using correct formula and showing all steps fully and leading to 1047675

(4)



(b) A sequence has n th term $u_n = \sin(90n^\circ)$, $n \geq 1$

(i) Find the order of the sequence.

$$u_1 = \sin(90^\circ) = 1, \quad u_2 = \sin(180^\circ) = 0,$$
$$u_3 = \sin(270^\circ) = -1, \quad u_4 = \sin(360^\circ) = 0, \quad u_5 = \sin(450^\circ) = 1, \dots$$

Order 4

B1 obtains the terms of this sequence are 1, 0, -1, 0, 1, 0, ... and deduces the order 4

(ii) Find $\sum_{r=1}^{222} u_r$ (1)

$\frac{222}{4} = 55.5$, so we need 55 and a half 'cycles' of the sequence

$$55(1 + 0 + -1 + 0) + 1 + 0 = 1$$

M1 for deducing that the sum can be found using the order of the sequence

A1 obtains 1

(2)

(Total for Question 4 is 7 marks)



5. $f(x) = \frac{1}{3}x^3 - 4x - 2$

- a. Show that the equation $f(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$, and state the values of the integers a and b .

$$\frac{1}{3}x^3 - 4x - 2 = 0 \Rightarrow x^3 = 12x + 6 \Rightarrow x^2 = 12 + \frac{6}{x} \Rightarrow x = \pm\sqrt{12 + \frac{6}{x}}$$

M1 equates $f(x) = 0$ and proceeds to an equation in $x = \pm\sqrt{\dots}$

A1 correct values of $a = 12, b = 6$

(2)

$f(x) = 0$ has one positive root, α .

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 4$ is used to find an approximation value for α .

- b. Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal place

$$x_1 = \sqrt{12 + \frac{6}{4}} = 3.6742, \quad x_2 = 3.6923, \quad x_3 = 3.6912, \quad x_4 = 3.6913$$

M1 Uses a formula $x = \sqrt{a + \frac{b}{x}}$ with 4 to find the value of x_1 correct to 4 d.p.

A1 correct values for $x_1 = 3.6742, x_2 = 3.6923, x_3 = 3.6912, x_4 = 3.6913$

(2)

- c. Explain why for this question, the Newton-Raphson method cannot be used with $x_1 = 2$.

B1 Accepts any reasonable reason why the Newton-Raphson method cannot be used with $x_1 = 2$

e.g. There is a stationary point at $x = 2$

Tangent to the curve would not meet the x -axis

Tangent to the curve is horizontal

(1)

(Total for Question 5 is 5 marks)



6. $f(x) = 2x^3 + 3x^2 - 1$

a. (i) Show that $(2x - 1)$ is a factor of $f(x)$.

(ii) Express $f(x)$ in the form $(2x - 1)(x + a)^2$ where a is an integer.

(i) If $(2x - 1)$ is a factor then by the factor theorem, $f\left(\frac{1}{2}\right) = 0$

$$\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 1 = 0 \quad \text{and so } (2x - 1) \text{ is a factor.}$$

B1 Correctly applies the factor theorem

$$(ii) \quad (2x - 1)(x^2 + 2ax + a^2) = 2x^3 + 3x^2 - 1$$

$$(2x^3 + 4ax^2 + 2a^2x - x^2 - 2ax - a^2) = 2x^3 + 3x^2 - 1$$

$$2x^3 + (4a - 1)x^2 + (2a^2 - 2a)x - a^2 = 2x^3 + 3x^2 - 1$$

$$(4a - 1) = 3 \Rightarrow a = 1, \quad (2a^2 - 2a) \Rightarrow a = 1$$

M1 Attempts to find the other quadratic factor $(x^2 \pm ax \pm b)$ by long division or equating coefficients

M1 factorises 3 terms quadratic to obtain $(x \pm a)^2$ e.g. $(x + 1)^2$

A1 correct answer only $(2x - 1)(x + 1)^2$

(4)

Using the answer to part a) (ii)

b. show that the equation $2p^6 + 3p^4 - 1$ has exactly two real solutions and state the values of these roots.

Let $x = p^2$, therefore the equation becomes

$$2x^3 + 3x^2 - 1 = 0$$

Using part ii), this factorises into $(2x - 1)(x + 1)^2 = 0 \Rightarrow x = \frac{1}{2}, x = -1$ (repeated root).

$$x = \frac{1}{2} \Rightarrow p^2 = \frac{1}{2} \Rightarrow p = \pm \frac{1}{\sqrt{2}}$$

$$x = -1 \Rightarrow p^2 = -1 \Rightarrow \text{no real roots}$$

So there are two real roots

M1 Uses part ii) to factorise and give a partial explanation that $p^2 = -1$ has no real roots

A1 Complete proof that the given equation has exactly two real solutions

(2)



c. deduce the number of real solutions, for $5\pi \leq \theta \leq 8\pi$, to the equation

$$2\cos^3\theta + 3\cos^2\theta - 1 = 0$$

$$\text{Let } y = \cos\theta \Rightarrow \cos\theta = -1, \frac{1}{2}.$$

$$\text{For } 5\pi \leq \theta \leq 8\pi, \cos\theta = -1 \Rightarrow \theta = 5\pi, 7\pi$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{17}{3}\pi, \frac{19}{3}\pi, \frac{23}{3}\pi$$

Therefore, there are 5 solutions

B1 deduces that there are 5 solutions

(1)

(Total for Question 6 is 7 marks)



7. i. Solve $0 \leq \theta \leq 180^\circ$, the equation

$$4 \cos \theta = \sqrt{3} \operatorname{cosec} \theta$$

$$4 \cos \theta = \frac{\sqrt{3}}{\sin \theta}$$

$$4 \cos \theta \sin \theta = \sqrt{3}$$

Using $\sin 2\theta = 2 \sin \theta \cos \theta$

$$2 \sin 2\theta = \sqrt{3} \Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{1}{2} \arcsin \left(\frac{\sqrt{3}}{2} \right) \Rightarrow \theta = 30^\circ, 60^\circ$$

B1 for $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$

M1 Applying $\sin 2\theta = 2 \sin \theta \cos \theta$ and proceeding to $\sin 2\theta = k$,

M1 Use the correct order of operations to find at least one value for θ in either radians or degrees

A1 both values are correct and no extra values in range.

(4)



(ii) Solve, for $0 \leq x \leq 2\pi$, the equation

$$\cos x - \sqrt{3} \sin x = \sqrt{3}$$

$$\cos x - \sqrt{3} \sin x = \sqrt{3} \Rightarrow R \cos(x - \alpha) = \sqrt{3}$$

$$R^2 = 1^2 + (-\sqrt{3})^2 = 4 \Rightarrow R = 2$$

$$\tan \alpha = \frac{-\sqrt{3}}{1} \Rightarrow \alpha = -\frac{\pi}{3}$$

$$2 \cos\left(x + \frac{\pi}{3}\right) = \sqrt{3}$$

$$x = \left(\cos^{-1} \frac{\sqrt{3}}{2}\right) - \frac{\pi}{3}$$

$$x = \frac{3\pi}{2}, \frac{11\pi}{6}$$

M1 expresses $\cos x - \sqrt{3} \sin x = \sqrt{3}$ in the form $R \cos(x \pm \alpha) = \sqrt{3}$

M1 uses $R \cos(x + \alpha)$ to find the values of both R and α

A1 $\cos\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

M1 Use the correct order of operations to find at least one value for x in either radians

or degrees $x = \arccos\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3}$

A1 correct solutions $x = \frac{3\pi}{2}, \frac{11\pi}{6}$

(5)

(Total for Question 7 is 9 marks)



8.

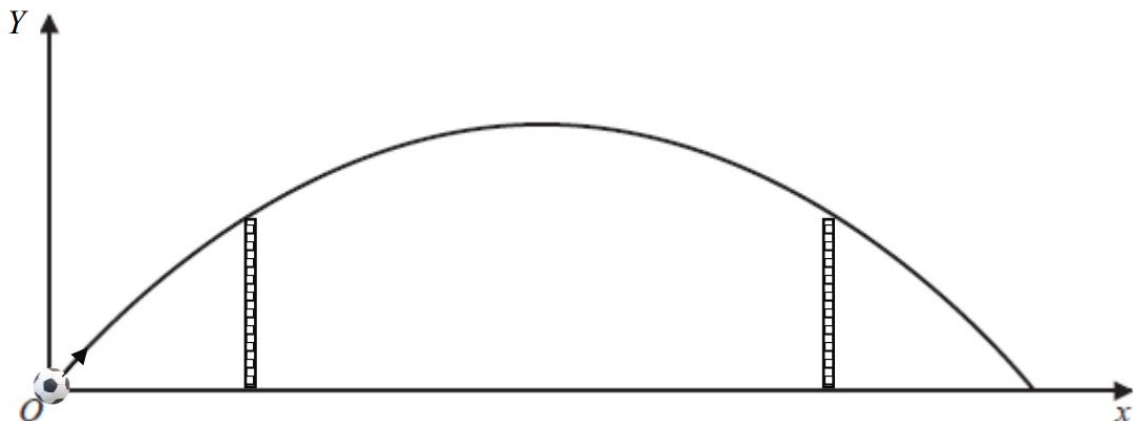


Figure 2

In a competition, competitors are going to kick a ball over the barrier walls. The height of the barrier walls are each 9 metres high and 50 cm wide and stand on horizontal ground.

The figure 2 is a graph showing the motion of a ball.

The ball reaches a maximum height of 12 metres and hits the ground at a point 80 metres from where its kicked.

a. Find a quadratic equation linking Y with x that models this situation.

The maximum is at (40,12) (due to symmetry) so using completing the square:

$$y = a(x - 40)^2 + 12$$

Substituting in (0,0):

$$\begin{aligned}
 0 &= a(-40)^2 + 12 \\
 -1600a &= 12 \Rightarrow a = \frac{-3}{400} \\
 y &= \frac{-3}{400}(x - 40)^2 + 12
 \end{aligned}$$

M1 translates the situation given into a suitable equation for the model.

M1 Applies a complete strategy with appropriate constraints to find all constants in their model.

A1 Finds a correct equation linking Y to x

$$Y = -\frac{3}{400}(x - 40)^2 + 12, \quad Y = -\frac{3}{400}x(x - 80)$$

(3)



The ball passes over the barrier walls.

b. Use your equation to deduce that the ball should be placed about 20 m from the first barrier wall.

$$9 = \frac{-3}{400}(x - 40)^2 + 12$$

$$(x - 40)^2 = 400$$

$$x = 20, x = 60$$

So the ball should be placed 20m from the first barrier wall.

M1 Substitutes $Y = 9$ into their quadratic equation and proceeds to obtain a term quadratic equation or a quadratic in the form $(x \pm c)^2 = d$

M1 Correct method of solving their quadratic equation to give at least one solution

A1 chooses the correct value for x i.e. 20

(3)

c. Give one limitation of the model.

B1 Gives a limitation of the model

- The ground is horizontal
- The ball needs to be kicked from the ground
- The ball is modelled as a particle
- There is no wind or air resistance on the ball
- There is no spin on the ball
- The trajectory of the ball is a perfect parabola

(1)

(Total for Question 8 is 7 marks)



9. Given that x is measured in radians, prove, from the first principles, that

$$\frac{d}{dx}(\sin x) = \cos x$$

You may assume the formula for $\sin(A \pm B)$ and that as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\sin h \cos x}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \end{aligned}$$

Using the limits provided in the question:

$$= \cos x$$

B1 Gives the correct fraction such as $\frac{\sin(x+h) - \sin x}{h}$

M1 uses the compound angle formula for $\sin(x + h)$ to give $\sin x \cos h \pm \cos x \sin h$

A1 Achieves $\frac{\sin x \cos h + \cos x \sin h}{h}$

M1 Complete attempt to apply the given limits to the gradient of their chord

A1 Correct solution only $\frac{d}{dx}(\sin x) = \cos x$

(5)

(Total for Question 9 is 5 marks)



10. Given that $y = 8$ at $x = 1$, solve the differential equation

$$\frac{dy}{dx} = \frac{(12x+9)y^{\frac{1}{3}}}{x}$$

Giving your answer in the form $y^2 = f(x)$.

Using separation of variables:

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int \frac{(12x+9)}{x} dx$$

$$\frac{3}{2}y^{\frac{2}{3}} = 12x + 9 \ln x + c$$

Substituting in $y = 8, x = 1$:

$$\frac{3}{2}(8)^{\frac{2}{3}} = 12(1) + 9 \ln(1) + (c)$$

$$c = -6$$

So,

$$y^2 = (8x + 6 \ln x - 4)^3$$

B1 Separates variables correctly. No need for integral signs

M1 Integrates left hand side $Ay^{-\frac{1}{3}}$ to give $By^{\frac{2}{3}}$

M1 Integrates the right hand side $A + \frac{B}{x}$ and obtains $Cx + D \ln x + (c)$

A1 Correct answer for lhs and rhs

M1 Substitutes $y = 8$ at $x = 1$ into their function and finds the value of the constant c and attempts to find $y^2 = \dots$.

A1 Correct solution only

(Total for Question 10 is 6 marks)



11. $\frac{-6x^2+24x-9}{(x-2)(1-3x)} \equiv A + \frac{B}{x-2} + \frac{C}{1-3x}$

a. Find the values of the constants A , B and C .

$$\frac{-6x^2 + 24x - 9}{(x - 2)(1 - 3x)} \equiv A + \frac{B}{x - 2} + \frac{C}{1 - 3x}$$

$$-6x^2 + 24x - 9 = A(x - 2)(1 - 3x) + B(1 - 3x) + C(x - 2)$$

Substituting $x = 2$:

$$-6(2)^2 + 24(2) - 9 \equiv A(2 - 2)(1 - 3(2)) + B(1 - 3(2)) + C((2) - 2)$$

$$15 = -5B \Rightarrow B = -3$$

Substituting $x = \frac{1}{3}$

$$-6\left(\frac{1}{3}\right)^2 + 24\left(\frac{1}{3}\right) - 9 \equiv A\left(\frac{1}{3} - 2\right)\left(1 - 3\left(\frac{1}{3}\right)\right) + B\left(1 - 3\left(\frac{1}{3}\right)\right) + C\left(\left(\frac{1}{3}\right) - 2\right)$$

$$-\frac{5}{3} = -\frac{5}{3}C \Rightarrow C = 1$$

Substituting $x = 1$ (can be any value)

$$-6(1)^2 + 24(1) - 9 = A(1 - 2)(1 - 3) - 3(1 - 3) + (1 - 2)$$

$$9 = 2A + 5 \Rightarrow A = 2$$

M1 Uses a correct identity

Either $-6x^2 + 24x - 9 \equiv A(x - 2)(1 - 3x) + B(1 - 3x) + C(x - 2)$

OR $10x - 5 \equiv B(1 - 3x) + C(x - 2)$

in a complete method to find values for B and C .

B1 $A = 2$

M1 Attempts to find the value of either B or C from their identity.

This can be achieved by either substituting values into their identity or by comparing coefficients and solving the resulting equations simultaneously

A1 $B = -3$ and $C = 1$

(4)



b. Using part (a), find $f'(x)$.

$$f(x) = 2 + \frac{-3}{x-2} + \frac{1}{1-3x}$$

$$f'(x) = \frac{3}{(x-2)^2} + \frac{3}{(1-3x)^2}$$

M1 Differentiates $2 - 3(x-2)^{-1} + (1-3x)^{-1}$ to give $A(x-2)^{-1} + B(1-3x)^{-1}$

A1 correct answer only $3(x-2)^{-2} + 3(1-3x)^{-2}$

(2)

c. Prove that $f(x)$ is an increasing function.

Both terms of $f'(x)$ are positive for all values of x , so $f(x)$ is an increasing function

A1 $f'(x) = +ve + +ve > 0$, so $f(x)$ is an increasing function

(1)

(Total for Question 11 is 7 marks)



12. a. Prove that

$$\frac{\sec^2 x - 1}{\sec^2 x} \equiv \sin^2 x$$

M1 either splits $\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\sec^2 x}{\sec^2 x} - \frac{1}{\sec^2 x}$ and uses $\frac{1}{\sec^2 x} = \cos^2 x$

or states or uses $\sec^2 x - 1 = \tan^2 x$ and $\sec^2 x = \frac{1}{\cos^2 x}$

M1 either replaces $1 - \cos^2 x = \sin^2 x$

or replaces $\frac{\tan^2 x}{\frac{1}{\cos^2 x}} = \frac{\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{1}$

A1 correct proof showing all necessary intermediate steps with no errors

(3)

b. Hence solve, for $-360^\circ < x < 360^\circ$, the equation

$$\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\cos 2x}{2}$$

Using part i) and the identity $\cos 2x = 1 - 2 \sin^2 x$:

$$\sin^2 x = \frac{1 - 2 \sin^2 x}{2}$$

$$4 \sin^2 x = 1$$

$$\sin x = \pm \frac{1}{2}$$

$$x = 30^\circ, 150^\circ, -210^\circ, -330^\circ, -30^\circ, -150^\circ, 210^\circ, 330^\circ$$

M1 Uses $\frac{\sec^2 x - 1}{\sec^2 x} = \sin^2 x$ and replaces $\cos 2x = 1 - 2 \sin^2 x$ and obtains an equation

of the form $A \sin^2 x = 1$

A1 Correct equation $4 \sin^2 x = 1$

A1 For two of $x = 30^\circ, 150^\circ, -210^\circ, -330^\circ, -30^\circ, -150^\circ, 210^\circ, 330^\circ$

A1 For $x = 30^\circ, 150^\circ, -210^\circ, -330^\circ, -30^\circ, -150^\circ, 210^\circ, 330^\circ$

(4)

(Total for Question 12 is 7 marks)



13. a. Find $\int \ln x \, dx$

Using integration by parts:

$$\begin{aligned}\int 1 \times \ln x &= x \ln x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + c\end{aligned}$$

M1 Uses integration by parts the correct way around

A1 Correct expression $x \ln x - \int x \cdot \frac{1}{x} \, dx$

M1 Integrates the second term $\int 1 \, dx = Dx \quad D \neq 0$

A1 correct integration with $+c$ $x \ln x - x + c$

(4)



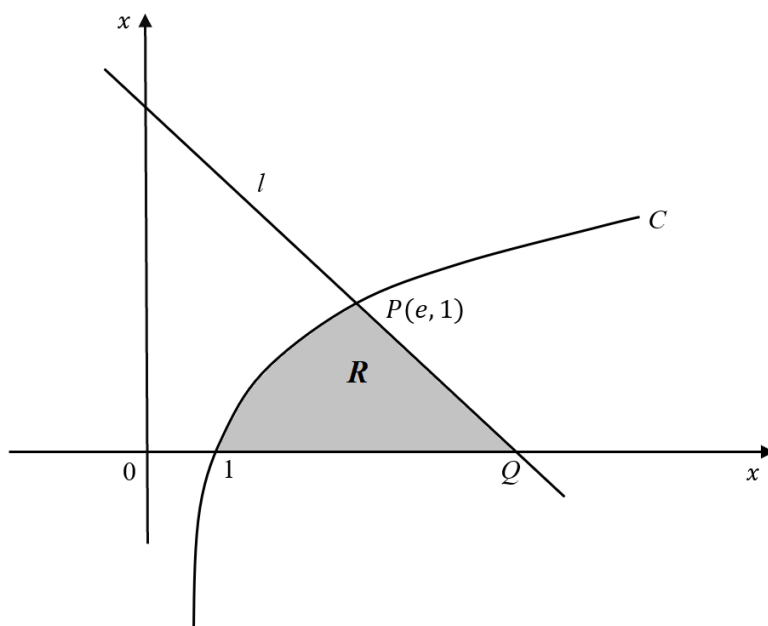


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = \ln x, \quad x > 0$$

The point P lies on C and has coordinate $(e, 1)$.

The line l is a normal to C at P . The line l cuts the x -axis at the point Q .

b. Find the exact value of the x -coordinate of Q .

$$\frac{dy}{dx} = \frac{1}{x}$$

Gradient of C at $(e, 1) = \frac{1}{e}, \Rightarrow$ Gradient of $l = -e$

$$l: y = -ex + c$$

Substituting in $(e, 1)$:

$$1 = -e(e) + c \Rightarrow c = 1 + e^2$$

Substituting $y = 0$ to find x -intercept:

$$0 = -ex + 1 + e^2 \Rightarrow x = \frac{e^2 + 1}{e}$$

B1 differentiates $\ln x$ to give $\frac{1}{x}$

M1 Complete strategy to find the x -coordinate where their normal to C at $P(e, 1)$ meets the x -axis

A1 l meets x -axis at $x = \frac{e^2+1}{e}$ or $e + \frac{1}{e}$

(3)



The finite region **R**, shown shaded in figure 3, is bounded by the curve, the line l and the x -axis.

c. Find the exact area of **R**.

$$\begin{aligned} R &= \int_1^e \ln x + \text{triangle } PQ(e, 0) \\ R &= \int_1^e \ln x + \frac{\frac{1}{e} \times 1}{2} \\ &= (e \ln e - e) - (1 \ln 1 - 1) + \frac{\frac{1}{e} \times 1}{2} \\ &= 1 + \frac{1}{2e} \end{aligned}$$

M1 Complete strategy of finding the area **R** by finding the sum of two key areas.

M1 Some evidence of applying limits of e and 1 and subtracts the correct way round

A1 correct solution only $1 + \frac{1}{2e}$

(3)

(Total for Question 13 is 10 marks)



14. A population of ants being studied on an island. The number of ants, P , in the population, is modelled by the equation.

$$P = \frac{900ke^{0.2t}}{1+ke^{0.2t}}, \text{ where } k \text{ is a constant.}$$

Given that there were 360 ants when the study started,

- a. show that $k = \frac{2}{3}$.

$$\text{At time } t = 0, P = 360 \Rightarrow 360 = \frac{900k}{1+k}$$

$$360(1+k) = 900k$$

$$360 + 360k = 900k \Rightarrow \frac{2}{3}$$

M1 setting $P = 360$ at $t = 0 \Rightarrow 360 = \frac{900k}{1+k}$

M1 obtains an equation in the form $360 = 540k$

A1 correct solution only $k = \frac{2}{3}$

(3)

- b. Show that $P = \frac{1800}{2+3e^{-0.2t}}$.

$$P = \frac{900 \times \frac{2}{3} e^{0.2t}}{1 + \frac{2}{3} e^{0.2t}} = \frac{1800e^{0.2t}}{3 + 2e^{0.2t}} = \frac{1800}{3e^{-0.2t} + 2}$$

B1 substitutes $k = \frac{2}{3}$ and divides both numerator and denominator by $e^{0.2t}$

i.e. $P = \frac{1800}{2+3e^{-0.2t}}$

(1)

The model predicts an upper limit to the number of ants on the island.

- c. State the value of this limit.

B1 sets $t \rightarrow \infty P = 900$

(1)



d. Find the value of t when $P = 520$. Give your answer to one decimal place.

$$520 = \frac{1800}{2 + 3e^{-0.2t}}$$

$$2 + 3e^{-0.2t} = \frac{1800}{520}$$

$$3e^{-0.2t} = \frac{19}{13}$$

$$e^{-0.2t} = \frac{19}{39}$$

$$-0.2t = \ln \frac{19}{39}$$

$$t = 3.5956 \dots$$

M1 substitutes $P = 520$ and proceeds to an equation of the form $Ae^{-0.2t} = B$

A1 correct equation $1560e^{-0.2t} = 760$ or equivalent

M1 correct order of operations using \ln 's to make t the subject.

A1 correct answer only $t = 3.6$

(4)



e. i. Show that the rate of growth, $\frac{dP}{dt} = \frac{P(900-P)}{4500}$

ii. Hence state the value of P at which the rate of growth is a maximum.

$$i. P = \frac{1800}{2+3e^{-0.2t}}$$

Using the quotient rule:

$$\frac{dP}{dt} = \frac{1800e^{-0.2t}}{(2+3e^{-0.2t})^2}$$

$$P = \frac{1800}{2+3e^{-0.2t}} \Rightarrow 2+3e^{-0.2t} = \frac{1800}{P} \Rightarrow e^{-0.2t} = \frac{\frac{1800}{P}-2}{3}$$

$$\frac{dP}{dt} = \frac{1800 \left(\frac{\frac{1800}{P}-2}{3} \right)}{\left(\frac{1800}{P} \right)^2} \frac{1800P-2P^2}{9000} = \frac{P(900-P)}{4500}$$

M1 attempts to differentiate using

A1 A correct differentiation statement

i.e. $1080e^{-0.2t}(2+3e^{-0.2t})^{-2}$ or $\frac{1080e^{-0.2t}}{(2+3e^{-0.2t})^2}$

M1 substitutes $2+3e^{-0.2t} = \frac{1800}{P}$ and $e^{-0.2t} = \frac{1800}{P} - 2$ or substitutes $e^{-0.2t} = \frac{\frac{1800}{P}-2}{3}$

into their $\frac{dP}{dt} = \dots$ to form an equation linking $\frac{dP}{dt}$ and P

A1 Correct algebra leading to $\frac{dP}{dt} = \frac{P(900-P)}{4500}$

ii. $\frac{d^2P}{dt^2} = 900 - 2P$

Rate of growth is at a maximum when $\frac{d^2P}{dt^2} = 0 \Rightarrow P = 450$

B1 450

(5)

(Total for Question 14 is 14 marks)

